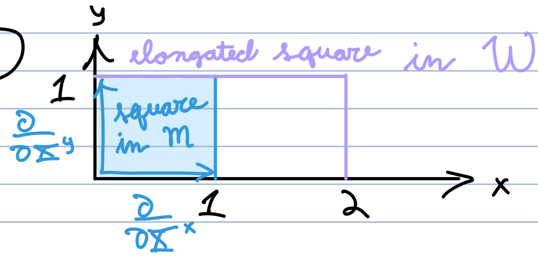
13. HW 3 : Elasticity

Continuous Theory

① 2D Square Elongation (youtube example)

– Calculate the deformation gradient F of a 2D square that has been elongated by twice its length along one of its dimensions.

①



$$\mathcal{M} \xrightarrow{\phi} \mathcal{W}$$

$$\frac{\partial}{\partial X^a} \qquad \frac{\partial}{\partial x^i}$$

$$\vec{x} = \phi(\vec{X}) = \begin{array}{c} \\ i\downarrow \end{array} \begin{bmatrix} 2X^x \\ X^y \end{bmatrix} \begin{array}{c} \frac{\partial}{\partial x^x} \\ \frac{\partial}{\partial x^y} \end{array}$$

$$F = \frac{\partial \phi^i}{\partial X^a} = i\downarrow \overset{a\,\longrightarrow}{\begin{bmatrix} \frac{\partial x^x}{\partial X^x} & \frac{\partial x^x}{\partial X^y} \\ \frac{\partial x^y}{\partial X^x} & \frac{\partial x^y}{\partial X^y} \end{bmatrix}}$$

$$= \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$$

② 3D Abstract Time-Dependent Motion (youtube ex.)

– Calculate F from the following flow map ϕ.

$$\phi(\vec{X}, t) = \begin{bmatrix} 3X^{x^2} X^y + tX^{z^3} \\ X^{z^2} - tX^x \\ 5X^y X^z \end{bmatrix}$$

② Assume :

$$\phi(\vec{X}, t) = \begin{bmatrix} 3X^{x^2} X^y + tX^{z^3} \\ X^{z^2} - tX^x \\ 5X^y X^z \end{bmatrix} i\downarrow$$

$$F = \frac{\partial \phi^i}{\partial X^a} = i\downarrow \overset{a\,\longrightarrow}{\begin{bmatrix} \frac{\partial x^x}{\partial X^x} & \frac{\partial x^x}{\partial X^y} & \frac{\partial x^x}{\partial X^z} \\ \frac{\partial x^y}{\partial X^x} & \frac{\partial x^y}{\partial X^y} & \frac{\partial x^y}{\partial X^z} \\ \frac{\partial x^z}{\partial X^x} & \frac{\partial x^z}{\partial X^y} & \frac{\partial x^z}{\partial X^z} \end{bmatrix}}$$

$$= \begin{bmatrix} 6X^x X^y & 3X^{x^2} & 3tX^{z^2} \\ -t & 0 & 2X^z \\ 0 & 5X^z & 5X^y \end{bmatrix}$$

- F may not be symmetric.
- F is a f^n of X and t.

Discrete Theory

① What are the steps involved in doing an elastic body simulation based on the Piola stress tensor framework?

Note: the subscript "c" means "per cell" and the subscript "v" means "per vertex".

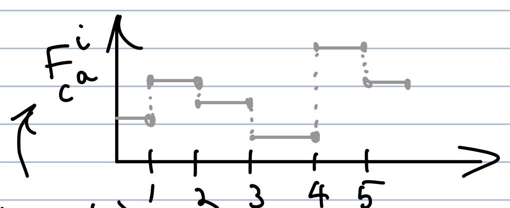
Equivalent approach:

$C_c \to \mathcal{U}_c \to \mathcal{U}_{tot} = \sum_c \mathcal{U}_c$

$f = \frac{\partial \mathcal{U}_{tot}}{\partial x_i}$

② Calculate F_c for a 2D mesh.

Note: We have a different F_c for each cell. Thus, $F^i_{c\,a}$ is piecewise constant.



(element of F_c)

① (i) $\phi(\vec{X}) = \vec{x}_v$ (flow map)

(ii) F_c (deformation grad.)

(iii) $C_c = F_c^T F_c$ (induced metric)

(iv) $E_c = \frac{1}{2}(C_c - I)$ (strain tensor)

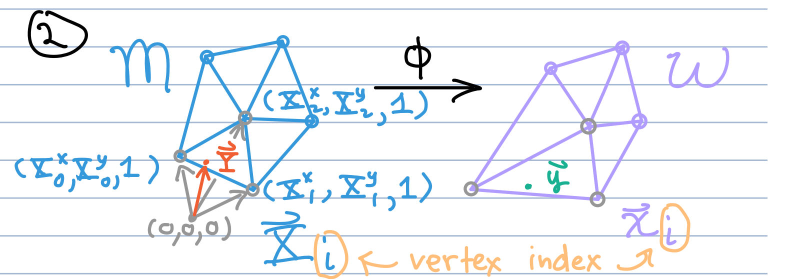
(v) Find appropriate stress-strain relation. One possible choice:
$S_c = 2\mu E_c + \lambda \, tr(E_c) I$ (2nd Piola stress)

(vi) $P_c = F S_c$ (1st Piola stress $= \frac{\partial U}{\partial F}$)

(vii) $\vec{f}_v = \frac{1}{n} \sum_{c>v} P_c \hat{n}_{c,v} A_{c,v}$
(= div(P_c) = discrete divergence)

(ix) $\dot{\vec{v}}_v = \vec{f}_v / m_v$, $\dot{\vec{X}}_v = \vec{v}_v$

②



$\vec{x}_i = \phi(\vec{X}_i)$ is the discrete (finite-dimensional) flow map.

$\vec{y} = \phi(\vec{Y})$ is the continuous (infinite-dimensional) flow map.

$F_c = \frac{\partial \phi^i}{\partial Y^a}$ (i ← increment over dim(W) = 2; a ← increment over dim(M) = 2)

Given $\begin{bmatrix} y^x \\ y^y \\ 1 \end{bmatrix} = \phi_c(\vec{Y}) = AB \begin{bmatrix} Y^x \\ Y^y \\ 1 \end{bmatrix}$

$$A = \begin{bmatrix} x^x_0 & x^x_1 & x^x_2 \\ x^y_0 & x^y_1 & x^y_2 \\ 1 & 1 & 1 \end{bmatrix}$$

$$B = \begin{bmatrix} X^x_0 & X^x_1 & X^x_2 \\ X^y_0 & X^y_1 & X^y_2 \\ 1 & 1 & 1 \end{bmatrix}^{-1}$$

$$Q = AB = \begin{bmatrix} Q^x_a & Q^x_b & Q^x_c \\ Q^y_a & Q^y_b & Q^y_c \\ Q^z_a & Q^z_b & Q^z_c \end{bmatrix}$$

we care about this part :

$$\therefore \; \phi_c(\vec{Y}) = \begin{bmatrix} Q_a^x Y^x + Q_b^x Y^y + Q_c^x \\ Q_a^y Y^x + Q_b^y Y^y + Q_c^y \\ Q_a^z Y^x + Q_b^z Y^y + Q_c^z \end{bmatrix} \downarrow i$$

$$F_c = \frac{\partial \phi_c^i}{\partial Y^a} = \begin{bmatrix} Q_a^x & Q_b^x \\ Q_a^y & Q_b^y \end{bmatrix}$$

(per cell)

$i = 0, 1$
$a = 0, 1$

Alternative approach for getting F_c (produces equivalent result):

$$F_c = -\frac{1}{nV_c} \sum_{j=0}^{n} \begin{bmatrix} | \\ x_j \\ | \end{bmatrix} \begin{bmatrix} - A_{c,j} n_{c,j}^T - \end{bmatrix}$$

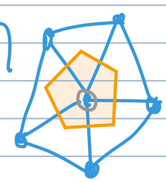
We use this to derive our discrete divergence operator in $\vec{f}_v = \mathrm{div}(P_c)$.

③ Calculate m_v.

Note: ρ is the mass density.

③ $$m_v = \rho \sum_{c \succ v} \frac{1}{n+1} V_c$$

m



M_v is a diagonal matrix (which makes matrix inversion easier).

References

(1) Chern, Albert. CSE 291 Course Notes, Spring 2024.

(2) Zubov, L. M. "Variational Principles of the Nonlinear Theory of Elasticity," Journal of Applied Mathematics and Mechanics, 35, 3, 406 – 410, 1971.